

# THE HARDY–LITTLEWOOD CONJECTURE AND RATIONAL POINTS

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## INTRODUCTION

The finite complexity case of the generalised Hardy–Littlewood conjecture recently proved by Green and Tao [GT10, GT12] and Green–Tao–Ziegler [GTZ12] is of fundamental importance to number theory. The aim of this note is to explore some of its consequences for the Hasse principle and weak approximation on algebraic varieties over  $\mathbb{Q}$ .

Hasse used Dirichlet’s theorem on primes in an arithmetic progression to deduce what is now called the Hasse principle for quadratic forms in four variables from the global reciprocity law and the Hasse principle for quadratic forms in three variables, itself a corollary of global class field theory (see [Hasse], p. 16 and p. 87). This ‘fibration method’ was taken up by Colliot-Thélène and Sansuc in [CSan82]. They showed that Schinzel’s Hypothesis (H), a vast generalisation of Dirichlet’s theorem in which  $at+b$  is replaced by a finite collection of irreducible polynomials in  $t$  of arbitrary degree [SchSi58], has strong consequences for the Hasse principle and weak approximation on conic bundles and some other pencils of varieties over  $\mathbb{Q}$ . The method of [CSan82] was extended to number fields and generalised in various directions by Serre (see [Serre], Ch. II, Annexe), Swinnerton-Dyer (see [Sw94], [Sw09] and references in that paper), Colliot-Thélène and others [CSw94, CSkSw98a, CSkSw98b, W07, Wei12]. In this note we show that when the ground field is  $\mathbb{Q}$  and the degenerate geometric fibres of the pencil are all defined over  $\mathbb{Q}$ , one can apply the methods of the aforementioned papers to obtain unconditional results by using the theorem of Green, Tao and Ziegler in place of Schinzel’s Hypothesis (H).

Additive combinatorics was recently applied to the study of rational points by Browning, Matthiesen and one of the authors in [BMS12]. The approach of [BMS12] uses the descent method of Colliot-Thélène and Sansuc; it crucially relies on the work of Matthiesen on representation of linear polynomials by binary quadratic forms [M12a, M12b] used to prove the Hasse principle and weak approximation for certain varieties appearing after descent. In this paper we give a short proof of most of the results of [BMS12] as well as a generalisation to certain equations involving norms of cyclic (and some non-cyclic) extensions. Our approach is based directly on the Green–Tao–Ziegler theorem and on various generalisations of the method of Hasse, Colliot-Thélène–Sansuc

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and Swinnerton-Dyer mentioned above, thus avoiding the use of descent and the results of [M12a, M12b].

We recall the theorem of Green, Tao and Ziegler and give its first corollaries in Section 1. In Section 2 we compare Schinzel's Hypothesis (H), in the form of Hypothesis (H<sub>1</sub>), with Proposition 2.1, a consequence of the Green–Tao–Ziegler theorem. We prove our main results and deduce their first applications in Section 3. In Section 4 we consider applications to representation of norms (and products of norms) by products of linear polynomials. An application of a different kind is given in Section 5: we prove that the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation for a family of del Pezzo surfaces of degree 4 of Picard rank 1, conditionally on the finiteness of the Tate–Shafarevich groups of elliptic curves of Mordell–Weil rank 0 over  $\mathbb{Q}$ .

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## 1. A COROLLARY OF THE GENERALISED HARDY–LITTLEWOOD CONJECTURE

In a series of papers Green and Tao [GT10, GT12] and Green–Tao–Ziegler [GTZ12] proved the generalised Hardy–Littlewood conjecture in the finite complexity case. The following qualitative statement is [GT10, Cor. 1.9].

**Theorem 1.1** (Green, Tao, Ziegler). *Let  $L_1(x, y), \dots, L_r(x, y) \in \mathbb{Z}[x, y]$  be pairwise non-proportional linear forms, and let  $c_1, \dots, c_r \in \mathbb{Z}$ . Assume that for each prime  $p$ , there exists  $(m, n) \in \mathbb{Z}^2$  such that  $p$  does not divide  $L_i(m, n) + c_i$  for any  $i = 1, \dots, r$ . Let  $K \subset \mathbb{R}^2$  be an open convex cone containing a point  $(m, n) \in \mathbb{Z}^2$  such that  $L_i(m, n) > 0$  for  $i = 1, \dots, r$ . Then there exist infinitely many pairs  $(m, n) \in K \cap \mathbb{Z}^2$  such that  $L_i(m, n) + c_i$  are all prime.*

We shall use the following easy corollary of this result. For a finite set of rational primes  $S$  we write  $\mathbb{Z}_S = \mathbb{Z}[S^{-1}]$ .

**Proposition 1.2.** *Suppose that we are given  $(\lambda_p, \mu_p) \in \mathbb{Q}_p^2$  for  $p$  in a finite set of primes  $S$ , and a positive real constant  $C$ . Let  $e_1, \dots, e_r$  be pairwise different elements of  $\mathbb{Z}_S$ . Then there exist infinitely many pairs  $(\lambda, \mu) \in \mathbb{Z}_S^2$  and pairwise different primes  $p_1, \dots, p_r$  not in  $S$  such that*

- (1)  $\lambda > C\mu > 0$ ;
- (2)  $(\lambda, \mu)$  is close to  $(\lambda_p, \mu_p)$  in the  $p$ -adic topology for  $p \in S$ ;
- (3)  $\lambda - e_i\mu = p_i u_i$ , where  $u_i \in \mathbb{Z}_S^*$ , for  $i = 1, \dots, r$ .

*Proof.* We can multiply  $\lambda, \mu$  and all  $\lambda_p, \mu_p$  by a product of powers of primes from  $S$ , and so assume without loss of generality that  $(\lambda_p, \mu_p) \in \mathbb{Z}_p^2$  for  $p \in S$ . Replacing  $C$  with a larger constant we assume  $C > e_i$ ,  $i = 1, \dots, r$ . Using the Chinese remainder theorem, we find  $\lambda_0 \in \mathbb{Z}$  such that  $\lambda_0 - \lambda_p$  is divisible by a sufficiently high power  $p^{n_p}$  for all  $p \in S$ , and similarly for  $\mu_0 \in \mathbb{Z}$ . In doing so we can assume that  $\lambda_0 > C\mu_0 > 0$ , in particular,  $\lambda_0 - e_i\mu_0 > 0$  for all  $i$ .

Let  $d$  be a product of powers of primes from  $S$  such that  $de_i \in \mathbb{Z}$  for  $i = 1, \dots, r$ . Let us write  $d(\lambda_0 - e_i\mu_0) = M_i c_i$ , where  $M_i$  is a product of powers of primes from  $S$ , and  $c_i \in \mathbb{Z}$  is coprime to the primes in  $S$ . Let  $N$  be a product of primes in  $S$  such that  $N > c_i - c_j$  for any  $i$  and  $j$ . Let  $M = \prod_{p \in S} p^{m_p}$  where

$$m_p \geq \max\{n_p, \text{val}_p(N) + \text{val}_p(M_i)\}, \quad i = 1, \dots, r.$$

Then  $N$  divides  $M/M_i$  for each  $i$ . We now look for  $\lambda$  and  $\mu$  of the form

$$\lambda = \lambda_0 + Mm, \quad \mu = \mu_0 + Mn, \quad (m, n) \in \mathbb{Z}^2. \quad (1.1)$$

Write  $L_i(x, y) = M_i^{-1}Md(x - e_iy)$ , then

$$\lambda - e_i\mu = d^{-1}M_i(L_i(m, n) + c_i) \quad (1.2)$$

for each  $i = 1, \dots, r$ . Let us check that the linear functions  $L_i(x, y) + c_i$  satisfy the condition of Theorem 1.1. For  $p \in S$ , the integer  $L_i(0, 0) + c_i$  is non-zero modulo  $p$  for each  $i$ . Now let  $p$  be a prime not in  $S$ . Since the determinant of the homogeneous part of the affine transformation (1.1) is in  $\mathbb{Z}_S^*$  and each  $M_i^{-1}d(\lambda - e_i\mu)$  equals  $M_i^{-1}d \in \mathbb{Z}_S^*$  at the point  $(\lambda, \mu) = (1, 0)$ , we see that there is  $(m, n) \in \mathbb{Z}^2$  such that  $\prod_{i=1}^r (L_i(m, n) + c_i)$  is not divisible by  $p$ .

We now choose an open convex cone  $K$ . Choose  $(m_0, n_0) \in \mathbb{Z}^2$ ,  $m_0 > Cn_0 > 0$ , for which the positive integers  $L_i(m_0, n_0)$  are pairwise different. After re-ordering the subscripts, we see that the inequalities

$$x > Cy > 0, \quad L_1(x, y) > \dots > L_r(x, y) > 0$$

hold for  $(x, y) = (m_0, n_0)$ . Define  $K \subset \mathbb{R}^2$  by these inequalities. We can apply Theorem 1.1 to the linear functions  $L_i(x, y) + c_i$  and the cone  $K$ . Thus there exist infinitely many pairs  $(m, n) \in K \cap \mathbb{Z}^2$  such that  $L_i(m, n) + c_i = p_i$ , where  $p_i$  is a prime not in  $S$ , for  $i = 1, \dots, r$ . The coefficients of each  $L_i(x, y)$  are divisible by  $N$ , hence

$$L_i(m, n) - L_{i+1}(m, n) \geq N > c_{i+1} - c_i.$$

Thus  $p_i > p_{i+1}$  for each  $i = 1, \dots, r-1$ , so all the primes  $p_i$  are pairwise different. Since  $n > 0$  and  $m > Cn$  we see that  $\mu = \mu_0 + Mn > 0$  and  $\lambda = \lambda_0 + Mm > C\mu$ .

By (1.2) we see that  $\lambda - e_i\mu$  differs from  $L_i(x, y) + c_i$  by an element of  $\mathbb{Z}_S^*$ , so the proof is now complete.  $\square$

Proposition 1.2 can be used to study the Hasse principle and weak approximation for rational points. In the proof of the following result, which is modelled on the original proof of Hasse (in the version of [CSan82, Prop. 2]), it replaces Dirichlet's theorem on primes in an arithmetic progression.

For a field extension  $K/\mathbb{Q}$  of degree  $n$  we denote by  $N_{K/\mathbb{Q}}(\mathbf{x})$  the corresponding norm form of degree  $n$  in  $n$  variables  $\mathbf{x} = (x_1, \dots, x_n)$ , defined by choosing a basis of the  $\mathbb{Q}$ -vector space  $K$ .

**Theorem 1.3.** *Let  $K_i$  be a cyclic extension of  $\mathbb{Q}$  of degree  $d_i$  and let  $b_i \in \mathbb{Q}^*$ ,  $e_i \in \mathbb{Q}$ , for  $i = 1, \dots, r$ . Then the quasi-affine variety  $V \subset \mathbb{A}^2 \times \mathbb{A}^{d_1} \times \dots \times \mathbb{A}^{d_r}$  over  $\mathbb{Q}$  defined by*

$$b_i(u - e_i v) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i) \neq 0, \quad i = 1, \dots, r, \quad (1.3)$$

*satisfies the Hasse principle and weak approximation.*

*Proof.* We are given  $M_p \in V(\mathbb{Q}_p)$  for each prime  $p$ , and  $M_0 \in V(\mathbb{R})$ . Let  $S$  be the set of places of  $\mathbb{Q}$  where we need to approximate. We include the real place in  $S$ . Note that the set of real points  $(u, v, \mathbf{x}_1, \dots, \mathbf{x}_r) \in V(\mathbb{R})$  for which  $(u, v) \in \mathbb{Q}^2$ , is dense in  $V(\mathbb{R})$ , and so it will be enough to prove the claim in the case when the coordinates  $u$  and  $v$  of  $M_0$  are in  $\mathbb{Q}$ . By a  $\mathbb{Q}$ -linear change of variables we can assume without loss of generality that  $M_0$  has coordinates  $(u, v) = (1, 0)$ . Then we have  $b_i > 0$  whenever  $K_i$  is totally imaginary.

We enlarge  $S$  so that  $b_i \in \mathbb{Z}_S^*$ ,  $e_i \in \mathbb{Z}_S$  and the field  $K_i$  is not ramified outside  $S$ , for all  $i = 1, \dots, r$ . Thus for each  $p \in S$  we now have a pair  $(\lambda_p, \mu_p) \in \mathbb{Q}_p^2$  such that

$$b_i(\lambda_p - e_i \mu_p) = N_{K_i/\mathbb{Q}}(\mathbf{x}_{i,p}) \neq 0, \quad i = 1, \dots, r,$$

for some  $\mathbf{x}_{i,p} \in (\mathbb{Q}_p)^{d_i}$ . Let  $C$  be a large positive constant to be specified later, such that  $C > e_i$  for  $i = 1, \dots, r$ . Let us remove repetitions from  $e_1, \dots, e_r$  and apply Proposition 1.2 to the resulting set of pairwise different elements of  $\mathbb{Z}_S^*$ . This produces  $(\lambda, \mu) \in \mathbb{Z}_S^2$ ,  $\lambda > C\mu > 0$ , such that for each  $i$  the number  $b_i(\lambda - e_i \mu)$  is a local norm for  $K_i/\mathbb{Q}$  at each finite place of  $S$ . This is also true for the real place because  $b_i > 0$  whenever  $K_i$  is totally imaginary, and  $\lambda - e_i \mu > 0$  for all  $i$ . Moreover, for each  $i$  we have  $b_i(\lambda - e_i \mu) = p_i u_i$ , where  $p_i$  is a prime not in  $S$ , and  $u_i \in \mathbb{Z}_S^*$  (we have  $p_i = p_j$  whenever  $e_i = e_j$ ).

Let  $(K_i/\mathbb{Q}, b_i(\lambda - e_i \mu)) \in \text{Br}(\mathbb{Q})$  be the class of the corresponding cyclic algebra. By continuity we have  $\text{inv}_p(K_i/\mathbb{Q}, b_i(\lambda - e_i \mu)) = 0$  for any  $p \in S$ , and also  $\text{inv}_{\mathbb{R}}(K_i/\mathbb{Q}, b_i(\lambda - e_i \mu)) = 0$ . Next,  $b_i(\lambda - e_i \mu)$  is a unit at every prime  $p \notin S \cup \{p_i\}$ , hence we obtain

$$\text{inv}_p(K_i/\mathbb{Q}, b_i(\lambda - e_i \mu)) = 0$$

for any  $p \neq p_i$ . The global reciprocity law now implies

$$\text{inv}_{p_i}(K_i/\mathbb{Q}, b_i(\lambda - e_i \mu)) = \text{inv}_{p_i}(K_i/\mathbb{Q}, p_i) = 0,$$

and since  $K_i/\mathbb{Q}$  is unramified outside  $S$ , the prime  $p_i$  splits completely in  $K_i$ . In particular,  $b_i(\lambda - e_i \mu)$  is a local norm at every place of  $\mathbb{Q}$ . By Hasse's theorem it is a global norm, so that

$$b_i(\lambda - e_i \mu) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i) \neq 0$$

for some  $\mathbf{x}_i \in \mathbb{Q}^{d_i}$ . This proves the Hasse principle for  $V$ .

Let us now prove weak approximation. Write  $d = d_1 \dots d_r$ . Using weak approximation in  $\mathbb{Q}$  we find a positive rational number  $\rho$  that is  $p$ -adically

close to 1 for each prime  $p \in S$ , and  $\rho^d$  is close to  $\lambda > 0$  in the real topology. We now make the change of variables

$$\lambda = \rho^d \lambda', \quad \mu = \rho^d \mu', \quad \mathbf{x}_i = \rho^{d/d_i} \mathbf{x}'_i, \quad i = 1, \dots, r.$$

Then  $(\lambda', \mu')$  is still close to  $(\lambda_p, \mu_p)$  in the  $p$ -adic topology for each prime  $p \in S$ . In the real topology  $(\lambda', \mu')$  is close to  $(1, \mu/\lambda)$ . Since  $0 < \mu/\lambda < C^{-1}$ , by choosing a large enough  $C$  we ensure that  $(\lambda', \mu')$  is close to  $(1, 0)$ . We can conclude by using weak approximation in the norm tori  $N_{K_i/\mathbb{Q}}(\mathbf{z}) = 1$ .  $\square$

**Remarks.** (1) In the case when all the fields  $K_i$  are quadratic this result is Thm. 1.2 of [BMS12]. Eliminating  $u$  and  $v$  one sees that  $V$  is then isomorphic to an open subset of a complete intersection of  $r - 2$  quadrics in  $\mathbb{A}^{2r}$  of a very special kind. These intersections of quadrics are important because of their relation to conic bundles, first pointed out by Salberger in [Sa86]. In [CSan87b] Colliot-Thélène and Sansuc proved the main results of descent theory and gave a description of universal torsors by explicit equations. Given the conclusion of Theorem 1.3 their results imply that the Brauer–Manin obstruction is the only obstruction to weak approximation on conic bundles over  $\mathbb{P}^1_{\mathbb{Q}}$  such that all the degenerate fibres are over  $\mathbb{Q}$ -points (this is Thm. 1.1 of [BMS12]; see [CSan87b], Thm. 2.6.4 and Chapter III, or [Sk01], Prop. 4.4.8 and Cor. 6.1.3).

(2) In Section 3.3 below we give a somewhat different proof of Theorem 1.3 deducing it from the main result of this paper, Theorem 3.1.

## 2. COMPARISON WITH SCHINZEL’S HYPOTHESIS (H)

Our aim in this section is to show that in the study of rational points on pencils of varieties Proposition 1.2 can replace Schinzel’s Hypothesis (H). In such applications it is often more convenient to use a consequence of (H), stated in [CSw94, Section 4] under the name of Hypothesis (H<sub>1</sub>). We reproduce it below in the case of linear polynomials over  $\mathbb{Q}$ . In this case it was already used in [CSan82, Section 5].

**Hypothesis (H<sub>1</sub>)** *Let  $e_1, \dots, e_r$  be pairwise different rational numbers. Let  $S$  be a finite set of primes containing the prime factors of the denominators of  $e_1, \dots, e_r$  and the primes  $p \leq r$ . Suppose that we are given  $\tau_p \in \mathbb{Q}_p$  for  $p \in S$  and a positive real constant  $C$ . Then there exist primes  $p_1, \dots, p_r$  not in  $S$  and  $\tau \in \mathbb{Q}$  such that*

- (1)  $\tau$  is arbitrarily close to  $\tau_p$  in the  $p$ -adic topology, for  $p \in S$ ;
- (2)  $\tau > C$ ;
- (3)  $\text{val}_p(\tau - e_i) = 0$  for any  $p \notin S \cup \{p_i\}$ ,  $i = 1, \dots, r$ ;
- (4)  $\text{val}_{p_i}(\tau - e_i) = 1$  for any  $i = 1, \dots, r$ .

Hypothesis (H<sub>1</sub>) is usually supplemented with the following statement. Let  $K/\mathbb{Q}$  be a cyclic extension unramified outside  $S$ . Assuming the conclusion of (H<sub>1</sub>) we have the following implication:

*if  $\sum_{p \in S} \text{inv}_p(K/\mathbb{Q}, \tau_p - e_i) = 0$  for some  $i$ , then  $p_i$  splits completely in  $K/\mathbb{Q}$ .*

Hypothesis (H<sub>1</sub>) and its supplement can be compared with the following consequence of Proposition 1.2.

**Proposition 2.1.** *Let  $e_1, \dots, e_r$  be pairwise different rational numbers. Let  $S$  be a finite set of primes containing the prime factors of the denominators of  $e_1, \dots, e_r$ . Suppose that we are given  $\tau_p \in \mathbb{Q}_p$  for  $p \in S$  and a positive real constant  $C$ . Then there exist pairwise different primes  $p_1, \dots, p_r$  not in  $S$  and  $\tau \in \mathbb{Q}$  such that*

- (1)  $\tau$  is arbitrarily close to  $\tau_p$  in the  $p$ -adic topology, for  $p \in S$ ;
- (2)  $\tau > C$ ;
- (3)  $\text{val}_p(\tau - e_i) \leq 0$  for any  $p \notin S \cup \{p_i\}$ ,  $i = 1, \dots, r$ ;
- (4)  $\text{val}_{p_i}(\tau - e_i) = 1$  for any  $i = 1, \dots, r$ ;
- (5) for any cyclic extension  $K/\mathbb{Q}$  unramified outside  $S$  and such that

$$\sum_{p \in S} \text{inv}_p(K/\mathbb{Q}, \tau_p - e_i) = c \in \mathbb{Q}/\mathbb{Z}$$

for some  $i$ , we have  $\text{inv}_{p_i}(K/\mathbb{Q}, \tau - e_i) = -c$ ; in particular, if  $c = 0$ , then  $p_i$  splits completely in  $K/\mathbb{Q}$ .

*Proof.* By increasing the list of  $e_i$ 's, we may assume  $r \geq 2$ . Let us also assume  $C > e_i$ ,  $i = 1, \dots, r$ . We then apply Proposition 1.2 to  $(\lambda_p, \mu_p) = (\tau_p, 1)$  for  $p \in S$ . This produces  $(\lambda, \mu) \in \mathbb{Z}_S^2$  such that  $\tau = \lambda/\mu$  satisfies all the properties in the proposition. Indeed, (1) and (2) are clear. For  $p \notin S$  we have  $\text{val}_p(\mu) \geq 0$ . For  $p \notin S \cup \{p_i\}$  we have  $\text{val}_p(\lambda - e_i\mu) = 0$ , so that

$$\text{val}_p(\tau - e_i) = \text{val}_p(\lambda - e_i\mu) - \text{val}_p(\mu) \leq 0,$$

which proves (3). From the assumption  $r \geq 2$  we can deduce that  $\text{val}_{p_i}(\mu) = 0$  for  $i = 1, \dots, r$ . Otherwise  $\text{val}_{p_i}(\mu) > 0$ , which would imply  $\text{val}_{p_i}(\lambda) > 0$ ; we would obtain  $\text{val}_{p_i}(\lambda - e_j\mu) > 0$  and therefore  $p_i = p_j$  for some  $j \neq i$ , thus contradicting the hypothesis that the primes  $p_1, \dots, p_r$  are pairwise different. This proves (4). Since  $(\lambda, \mu)$  is close to  $(\tau_p, 1)$  in the  $p$ -adic topology for  $p \in S$ , by continuity we have

$$\sum_{p \in S} \text{inv}_p(K/\mathbb{Q}, \lambda - e_i\mu) = c.$$

We also have  $\lambda - e_i\mu > 0$ , hence  $\text{inv}_{\mathbb{R}}(K/\mathbb{Q}, \lambda - e_i\mu) = 0$ . By the global reciprocity law of class field theory this implies

$$\sum_{p \notin S} \text{inv}_p(K/\mathbb{Q}, \lambda - e_i\mu) = -c.$$

Since  $K/\mathbb{Q}$  is unramified outside  $S$ , we have  $\text{inv}_p(K/\mathbb{Q}, \lambda - e_i\mu) = 0$  for any prime  $p \notin S \cup \{p_i\}$ , because in this case  $\text{val}_p(\lambda - e_i\mu) = 0$ . Thus

$$\text{inv}_{p_i}(K/\mathbb{Q}, \lambda - e_i\mu) = -c.$$

In the case when  $c = 0$ , we deduce from  $\text{val}_{p_i}(\lambda - e_i\mu) = 1$  that  $p_i$  splits completely in  $K$ , so that (5) is proved.  $\square$



**Remark.** One can give a stronger variant of this proposition by proving that  $\tau$ , in addition to properties (1) to (5), can be chosen in a given Hilbertian subset of  $\mathbb{Q}$ . The proof uses Ekedahl’s effective version of Hilbert’s irreducibility theorem and the fact that any Hilbertian subset of  $\mathbb{Q}$  is open in the topology induced by the product topology of  $\prod_p \mathbb{Q}_p$ , where the product is over all primes. We do not give a detailed proof, because this variant will not be used in this paper.

### 3. VARIETIES FIBRED OVER THE PROJECTIVE LINE

**3.1. Main results.** Recall that for a variety  $X$  over  $\mathbb{Q}$  one denotes the image of  $\text{Br}(\mathbb{Q})$  in  $\text{Br}(X)$  by  $\text{Br}_0(X)$ . If  $\pi : X \rightarrow \mathbb{P}^1$  is a dominant morphism of integral varieties over  $\mathbb{Q}$ , then the corresponding vertical Brauer group is defined as follows:

$$\text{Br}_{\text{vert}}(X) = \text{Br}(X) \cap \pi^* \text{Br}(\mathbb{Q}(\mathbb{P}^1)) \subset \text{Br}(\mathbb{Q}(X)).$$

By a  $\mathbb{Q}$ -fibre of  $\pi : X \rightarrow \mathbb{P}^1$  we understand a fibre above a  $\mathbb{Q}$ -point of  $\mathbb{P}^1$ .

We denote the completions of  $\mathbb{Q}$  by  $\mathbb{Q}_v$ , and the ring of adèles of  $\mathbb{Q}$  by  $\mathbf{A}_{\mathbb{Q}}$ . Let  $\overline{\mathbb{Q}}$  be an algebraic closure of  $\mathbb{Q}$ . For a subfield  $K \subset \overline{\mathbb{Q}}$  we write  $\Gamma_K$  for the Galois group  $\text{Gal}(\overline{\mathbb{Q}}/K)$ .

**Theorem 3.1.** *Let  $X$  be a geometrically integral variety over  $\mathbb{Q}$  with a smooth and surjective morphism  $\pi : X \rightarrow \mathbb{P}^1$  such that*

(a) *with the exception of finitely many  $\mathbb{Q}$ -fibres, denoted by  $X_1, \dots, X_r$ , each fibre of  $\pi$  contains a geometrically integral irreducible component;*

(b) *for each  $i$ , the fibre  $X_i$  contains an irreducible component such that the algebraic closure of  $\mathbb{Q}$  in its field of functions is an abelian extension of  $\mathbb{Q}$ .*

*Then  $\mathbb{P}^1(\mathbb{Q}) \cap \pi(X(\mathbf{A}_{\mathbb{Q}}))$  is dense in  $\pi(X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}}) \subset \mathbb{P}^1(\mathbf{A}_{\mathbb{Q}}) = \prod_v \mathbb{P}^1(\mathbb{Q}_v)$ .*

Note that the assumptions of Theorem 3.1 imply that the generic fibre of  $\pi : X \rightarrow \mathbb{P}^1$  is geometrically integral. Thus all but finitely many  $\mathbb{Q}$ -fibres of  $\pi$  are geometrically integral. The cokernel of the natural map  $\text{Br}(\mathbb{Q}) \rightarrow \text{Br}_{\text{vert}}(X)$  is finite by [CSk00, Lemma 3.1]. Note finally that when  $r = 1$ , the statement of Theorem 3.1 is well known, see Section 2.1 of [CSkSw98a].

*Proof of Theorem 3.1.* Without loss of generality we can assume that  $X_i$  is the fibre above a point  $e_i \in \mathbb{A}^1(\mathbb{Q})$ , for  $i = 1, \dots, r$ . Let  $K_i$  be the abelian extension of  $\mathbb{Q}$  as in (b).

We follow the proof of [CSkSw98a, Thm. 1.1] which uses the same method as [CSw94, Thm. 4.2]. Steps 1 and 2 of the proof below repeat the proof of [CSkSw98a, Thm. 1.1] *verbatim*. Step 3 contains modifications necessary to replace the use of Hypothesis (H<sub>1</sub>) by Proposition 2.1.

*Step 1.* Let us recall a well-known description of  $\text{Br}_{\text{vert}}(X)$ . Write  $K_i$  as a compositum of cyclic extensions  $K_{ij}/\mathbb{Q}$ , and let  $\chi_{ij} : \Gamma_{\mathbb{Q}} \rightarrow \mathbb{Q}/\mathbb{Z}$  be a character such that  $K_{ij}$  is isomorphic to the invariant subfield of  $\text{Ker}(\chi_{ij})$ . Let  $t$  be a coordinate on  $\mathbb{A}^1 \subset \mathbb{P}^1$  so that  $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$ . The class

$$A_{ij} = (K_{ij}/\mathbb{Q}, t - e_i) \in \text{Br}(\mathbb{Q}(t))$$

of the corresponding cyclic algebra is ramified on  $\mathbb{P}^1$  only at  $e_i$  and  $\infty$  with residues  $\chi_{ij}$  and  $-\chi_{ij}$ , respectively. Let  $A \in \text{Br}(\mathbb{Q}(t))$  be such that  $\pi^*A \in \text{Br}(X)$ . Assumptions (a) and (b), together with [CSw94, Prop. 1.1.1], imply that  $A$  on  $\mathbb{P}^1$  is unramified away from  $e_1, \dots, e_r$ , and that the residue of  $A$  at  $e_i$  belongs to

$$\text{Ker}[\text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\Gamma_{K_i}, \mathbb{Q}/\mathbb{Z})].$$

This group is generated by the characters  $\chi_{ij}$ . Hence there exist  $n_{ij} \in \mathbb{Z}$  such that  $A - \sum n_{ij}A_{ij}$  is unramified on  $\mathbb{A}^1$ . Since  $\text{Br}(\mathbb{A}^1) = \text{Br}(\mathbb{Q})$  we conclude that  $A = \sum n_{ij}A_{ij} + A_0$  for some  $A_0 \in \text{Br}(\mathbb{Q})$ , and this implies, by considering residues at  $\infty$ , that

$$\sum n_{ij}\chi_{ij} = 0 \in \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z}). \quad (3.1)$$

Therefore, every element of  $\text{Br}_{\text{vert}}(X)$  is of the form  $\sum n_{ij}\pi^*A_{ij} + A_0$  for some  $n_{ij}$  satisfying (3.1) and some  $A_0 \in \text{Br}(\mathbb{Q})$ .

*Step 2.* We can clearly assume that  $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}} \neq \emptyset$ , otherwise there is nothing to prove. Pick any  $(M_p) \in X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}}$ , where  $M_0$  is a point in  $X(\mathbb{R})$ . By a small deformation we can assume that  $M_p$  does not belong to any of the fibres  $X_1, \dots, X_r$ .

We include the real place in the finite set of places  $S$  where we need to approximate. The set of real points  $M_0 \in V(\mathbb{R})$  for which  $\pi(M_0) \in \mathbb{P}^1(\mathbb{Q})$  is dense in  $V(\mathbb{R})$ , and so it is enough to approximate adelic points  $(M_p)$  such that  $\pi(M_0) \in \mathbb{P}^1(\mathbb{Q})$ . By a change of variables we then assume that  $\pi(M_0) = \infty$ . By another small deformation of the points  $M_p$  for each prime  $p$  we can further assume that  $\pi(M_p) \neq \infty$  when  $p \neq 0$ .

We include in  $S$  the primes of bad reduction for  $X$ . We ensure that  $e_i \in \mathbb{Z}_S$  for each  $i = 1, \dots, r$ ,  $e_i - e_j \in \mathbb{Z}_S^*$  for all  $i \neq j$ , and no prime outside of  $S$  is ramified in any of the fields  $K_i$ . Furthermore, we increase  $S$  so that if  $K_i$  has a place of degree 1 over  $p \notin S$ , then the corresponding  $\mathbb{F}_p$ -component of the degenerate fibre of  $\pi$  over the reduction of  $e_i$  has an  $\mathbb{F}_p$ -point. This is achieved by using the Lang–Weil estimate, see [CSkSw98a, Lemma 1.2]. By a similar argument we assume that on the reduction of  $X$  modulo  $p \notin S$  any geometrically integral component of a fibre over an  $\mathbb{F}_p$ -point contains an  $\mathbb{F}_p$ -point. All these  $\mathbb{F}_p$ -points are smooth, because  $\pi$  is a smooth morphism.

Since  $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}} \neq \emptyset$ , by the result of Step 1 we can use Harari’s ‘formal lemma’ [H94, Cor. 2.6.1] to increase  $S \subset S_1$  and choose  $M_p \in X(\mathbb{Q}_p)$  for  $p \in S_1 \setminus S$  away from the fibres  $X_1, \dots, X_r$  so that for all  $i, j$  we have

$$\sum_{p \in S_1} \text{inv}_p(A_{ij}(\pi(M_p))) = 0. \quad (3.2)$$

*Step 3.* Let  $\tau_p$  be the coordinate of  $\pi(M_p)$ , where  $p$  is a prime in  $S_1$ . An application of Proposition 2.1 produces  $\tau \in \mathbb{Q}$  which is an arbitrarily large positive real number, and is close to  $\tau_p$  in the  $p$ -adic topology for the primes  $p \in S_1$ .



Let us prove that  $X_\tau(\mathbf{A}_\mathbb{Q}) \neq \emptyset$ . By the inverse function theorem we have  $X_\tau(\mathbb{R}) \neq \emptyset$  and  $X_\tau(\mathbb{Q}_p) \neq \emptyset$  for  $p \in S_1$ . Thus it remains to consider the following two cases.

$\mathbb{Q}_v = \mathbb{Q}_p$ , where  $p = p_i$ ,  $i = 1, \dots, r$ . Since  $\text{val}_{p_i}(\tau - e_i) = 1$ , the reduction of  $\tau$  modulo  $p_i$  equals the reduction of  $e_i$ . In view of (3.2) property (5) of Proposition 2.1 implies that for each given value of  $i$  all the cyclic fields  $K_{ij}$  are split at  $p_i$ , and thus  $K_i$  is also split. Hence there is a geometrically integral irreducible component of the  $\mathbb{F}_{p_i}$ -fibre over the reduction of  $e_i$  modulo  $p_i$ . We arranged that it has an  $\mathbb{F}_{p_i}$ -point. By Hensel's lemma it gives rise to a  $\mathbb{Q}_{p_i}$ -point in  $X_\tau$ .

$\mathbb{Q}_v = \mathbb{Q}_p$ , where  $p \notin S_1 \cup \{p_1, \dots, p_r\}$ . We have  $\text{val}_p(\tau - e_i) \leq 0$  for each  $i = 1, \dots, r$ , and hence the reduction of  $\tau$  modulo  $p$  is a point of  $\mathbb{P}^1(\mathbb{F}_p)$  other than the reduction of any of  $e_1, \dots, e_r$ . Thus any  $\mathbb{F}_p$ -point on a geometrically integral irreducible component of the fibre at  $\tau \bmod p$  gives rise to a  $\mathbb{Q}_p$ -point on  $X_\tau$ , by Hensel's lemma.

In both cases we constructed a  $\mathbb{Q}_p$ -point that comes from a  $\mathbb{Z}_p$ -point on an integral model of  $X_\tau$ , therefore  $X_\tau(\mathbf{A}_\mathbb{Q}) \neq \emptyset$ . The theorem is proved.  $\square$

**Corollary 3.2.** *In the situation of Theorem 3.1, let us further assume that all but finitely many  $\mathbb{Q}$ -fibres of  $\pi : X \rightarrow \mathbb{P}^1$  satisfy the Hasse principle. Then  $\pi(X(\mathbb{Q}))$  is dense in  $\pi(X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}})$ . If, in addition, these  $\mathbb{Q}$ -fibres  $X_\tau$  are such that  $X_\tau(\mathbb{Q})$  is dense in  $X_\tau(\mathbf{A}_\mathbb{Q})$ , then  $X(\mathbb{Q})$  is dense in  $X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}}$ .*

**Remark.** If the generic fibre of  $\pi : X \rightarrow \mathbb{P}^1$  is proper, then all but finitely many fibres of  $\pi$  are proper. For proper  $\mathbb{Q}$ -fibres  $X_\tau$  the approximation assumption in Corollary 3.2 is that of *weak approximation*, since in this case  $X_\tau(\mathbf{A}_\mathbb{Q}) = \prod_v X_\tau(\mathbb{Q}_v)$ . By Hironaka's theorem one can always replace  $\pi : X \rightarrow \mathbb{P}^1$  by a partial compactification  $\pi' : X' \rightarrow \mathbb{P}^1$  such that  $X$  is a dense open subset of  $X'$  and the morphism  $\pi'$  is smooth with proper generic fibre.

We now give a statement for a smooth and proper variety  $X$ , to be compared with [CSkSw98a, Thm. 1.1].

**Theorem 3.3.** *Let  $X$  be a smooth, proper and geometrically integral variety over  $\mathbb{Q}$  with a surjective morphism  $\pi : X \rightarrow \mathbb{P}^1$  such that*

- (a) *with the exception of finitely many  $\mathbb{Q}$ -fibres, denoted by  $X_1, \dots, X_r$ , each fibre of  $\pi$  contains a geometrically integral irreducible component of multiplicity one;*
- (b) *for each  $i$ , the fibre  $X_i$  contains an irreducible component of multiplicity one such that the algebraic closure of  $\mathbb{Q}$  in its field of functions is an abelian extension of  $\mathbb{Q}$ .*

*Then  $\mathbb{P}^1(\mathbb{Q}) \cap \pi(X(\mathbf{A}_\mathbb{Q}))$  is dense in  $\pi(X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}}) \subset \mathbb{P}^1(\mathbf{A}_\mathbb{Q}) = \prod_v \mathbb{P}^1(\mathbb{Q}_v)$ . If, moreover, all but finitely many  $\mathbb{Q}$ -fibres of  $\pi$  satisfy the Hasse principle and weak approximation, then  $X(\mathbb{Q})$  is dense in  $X(\mathbf{A}_\mathbb{Q})^{\text{Brvert}}$ .*

*Proof.* Let  $Y \subset X$  denote the smooth locus of  $\pi$ . By (a) and (b) each fibre of  $\pi : X \rightarrow \mathbb{P}^1$  contains a multiplicity one irreducible component, hence  $\pi(Y) =$

$\mathbb{P}^1$ . Thus Theorem 3.1 can be applied to  $\pi : Y \rightarrow \mathbb{P}^1$ . We have  $\mathrm{Br}_{\mathrm{vert}}(X) = \mathrm{Br}(X) \cap \mathrm{Br}_{\mathrm{vert}}(Y)$ , and by [CSk00, Lemma 3.1], the set  $Y(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}_{\mathrm{vert}}(Y)}$  is dense in  $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}_{\mathrm{vert}}(X)}$ . This proves the first statement. The second one now follows from Corollary 3.2.  $\square$

**Remarks.** (1) We note that assumptions (a) and (b) will hold for any smooth, proper and geometrically integral variety  $X'$  over  $\mathbb{Q}$  with a surjective morphism  $\pi' : X' \rightarrow \mathbb{P}^1$  such that the generic fibres of  $\pi$  and  $\pi'$  are isomorphic. This follows from [Sk96, Lemma 1.1], see also [W07, Lemme 3.8].

(2) Using the remark after Proposition 2.1 one can prove a stronger variant of Theorem 3.3 with “all but finitely many  $\mathbb{Q}$ -fibres of  $\pi$ ” replaced by “the  $\mathbb{Q}$ -fibres of  $\pi$  over a Hilbertian subset of  $\mathbb{Q}$ ”. The details are left to the interested reader.

### 3.2. Application to pencils of Severi–Brauer and similar varieties.

Theorem 3.3 can be applied to the varieties considered by Colliot-Thélène and Swinnerton-Dyer in [CSw94], see also [CSkSw98a].

**Corollary 3.4.** *Let  $X$  be a smooth, proper and geometrically integral variety over  $\mathbb{Q}$  with a morphism  $\pi : X \rightarrow \mathbb{P}^1$ . Suppose that the generic fibre of  $\pi$  is a Severi–Brauer variety (for example, a conic), a 2-dimensional quadric, or a product of such. If all the fibres of  $\pi$  that are not geometrically integral are  $\mathbb{Q}$ -fibres, then  $X(\mathbb{Q})$  is dense in  $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}_{\mathrm{vert}}}$ .*

*Proof.* The assumptions of Theorem 3.3 are satisfied for  $X$  and  $\pi$  by well-known results of class field theory and by the structure of fibres of regular models of quadric bundles [Sk90] and Artin models of Severi–Brauer varieties [F97] (or see [W07, pp. 117–118] for an alternative argument).  $\square$

In particular, this result gives a uniform approach to Theorems 1.1, 1.3 and 1.4 of [BMS12] that does not use descent and leads to shorter and more natural proofs. See [BMS12] for a survey of known results on conic and quadric bundles, most of which are established over an arbitrary number field in place of  $\mathbb{Q}$  but under a strong restriction on the number of degenerate geometric fibres.

**3.3. Theorem 1.3 as a consequence of Theorem 3.1.** Here is one way to deduce Theorem 1.3 from Corollary 3.2 that keeps the Brauer group calculations to the minimum.

Let  $W$  be the quasi-affine subvariety of  $\mathbb{A}^2 \times \mathbb{A}^{d_1} \times \dots \times \mathbb{A}^{d_r}$  given by

$$b_i(u - e_i v) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i), \quad i = 1, \dots, r, \quad (u, v) \neq (0, 0).$$

The variety  $V$  defined in (1.3) is a dense open subset of  $W$ . The projection to the coordinates  $(u, v)$  defines a morphism  $W \rightarrow \mathbb{A}^2 \setminus (0, 0)$ . Let  $\pi : W \rightarrow \mathbb{P}^1$  be the composed morphism  $W \rightarrow \mathbb{A}^2 \setminus (0, 0) \rightarrow \mathbb{P}^1$ , and let  $X \subset W$  be the smooth locus of  $\pi$ . It is easy to see that  $\pi(X) = \mathbb{P}^1$ . Let  $\pi' : Y \rightarrow \mathbb{P}^1$  be a partial compactification of  $\pi : X \rightarrow \mathbb{P}^1$  as in the remark after Corollary 3.2, so that  $\pi'$  is smooth with proper generic fibre.

Let  $t = u/v$  be a coordinate on  $\mathbb{P}^1$ . It is straightforward to see that conditions (a) and (b) of Theorem 3.1 hold. In order to deduce Theorem 1.3 from Corollary 3.2 we need to prove that

- (1) geometrically integral, proper  $\mathbb{Q}$ -fibres of  $\pi'$  satisfy the Hasse principle and weak approximation;
- (2)  $\text{Br}_{\text{vert}}(X) = \text{Br}_0(X)$ .

The fibre of  $\pi$  at  $\tau \in \mathbb{Q}$ ,  $\tau \neq e_i$ , is the affine variety

$$\frac{N_{K_1/\mathbb{Q}}(\mathbf{x}_1)}{b_1(\tau - e_1)} = \dots = \frac{N_{K_r/\mathbb{Q}}(\mathbf{x}_r)}{b_r(\tau - e_r)} \neq 0.$$

This is a principal homogeneous space of the torus  $T$  defined by

$$N_{K_1/\mathbb{Q}}(\mathbf{t}_1) = \dots = N_{K_r/\mathbb{Q}}(\mathbf{t}_r) \neq 0.$$

The Hasse principle and weak approximation hold for smooth compactifications of principal homogeneous spaces of  $T$  if  $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}) = 0$ , see [San81, Ch. 8] or [Sk01, Thm. 6.3.1]. Let  $T_i = R_{K_i/\mathbb{Q}}^1(\mathbb{G}_m)$  be the norm torus attached to  $K_i/\mathbb{Q}$ , and let  $G_i$  be the cyclic group  $\text{Gal}(K_i/\mathbb{Q})$ . Then  $T$  is an extension of  $\mathbb{G}_m$  by the product of the tori  $T_i$ , so that we have the exact sequence of  $\Gamma_{\mathbb{Q}}$ -modules of characters

$$0 \rightarrow \mathbb{Z} \rightarrow \hat{T} \rightarrow \prod_{i=1}^r \hat{T}_i \rightarrow 0. \quad (3.3)$$

The long exact sequence of Galois cohomology gives rise to the exact sequence

$$\prod_{i=1}^r \text{Hom}(G_i, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(\mathbb{Q}, \hat{T}) \rightarrow \prod_{i=1}^r H^2(\mathbb{Q}, \hat{T}_i).$$

Let  $K$  be the compositum of the fields  $K_i$ , and let  $G = \text{Gal}(K/\mathbb{Q})$ . Since  $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}_i) = 0$  it is enough to show that if  $\alpha \in \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z})$  goes to  $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T})$  in  $H^2(\mathbb{Q}, \hat{T})$ , then  $\alpha \in \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ . But the tori  $T_i$  and  $T$  are split by  $K$ , hence (3.3) is split as an extension of  $\Gamma_K$ -modules. It follows that the restriction of  $\alpha$  to  $H^2(K, \mathbb{Z}) = \text{Hom}(\Gamma_K, \mathbb{Q}/\mathbb{Z})$  is in  $\text{III}_{\omega}^2(K, \mathbb{Z}) = 0$ , thus  $\alpha \in \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ . This proves (1).

Let us prove (2). Write  $\chi_i \in \text{Hom}(\Gamma_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z})$  for the image of a generator of  $\text{Hom}(G_i, \mathbb{Q}/\mathbb{Z})$ . Recall from Step 1 of the proof of Theorem 3.1 that any  $A \in \text{Br}(\mathbb{Q}(t))$  such that  $\pi^*A \in \text{Br}(X)$  can be written as

$$A = \sum_{i=1}^r n_i(\chi_i, t - e_i) + A_0, \quad \text{where} \quad \sum_{i=1}^r n_i \chi_i = 0,$$

for some  $A_0 \in \text{Br}(\mathbb{Q})$ . In  $\text{Br}(X)$  we have

$$\pi^*A = \sum_{i=1}^r n_i(\chi_i, u - e_i v) - \sum_{i=1}^r n_i(\chi_i, v) + A_0 = - \sum_{i=1}^r n_i(\chi_i, b_i) + A_0 \in \text{Br}_0(X),$$

since  $(\chi_i, N_{K_i/\mathbb{Q}}(\mathbf{x}_i)) = 0$  in  $\text{Br}(\mathbb{Q}(X))$ . This proves (2).

## 4. NORMS AND THEIR PRODUCTS AS PRODUCTS OF LINEAR POLYNOMIALS

**4.1. Cyclic extensions.** Consider the following system of Diophantine equations:

$$N_{K_i/\mathbb{Q}}(\mathbf{x}_i) = P_i(t), \quad i = 1, \dots, r, \quad (4.1)$$

where  $K_i/\mathbb{Q}$  are cyclic extensions and the polynomials  $P_i(t)$  are products of (possibly repeated) linear factors over  $\mathbb{Q}$ .

**Corollary 4.1.** *Let  $X$  be a smooth, proper and geometrically integral variety over  $\mathbb{Q}$  with a morphism  $\pi : X \rightarrow \mathbb{P}^1$  such that the generic fibre of  $\pi$  is birationally equivalent to the affine variety (4.1) over  $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$ . Then  $X(\mathbb{Q})$  is dense in  $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}_{\text{vert}}}$ .*

*Proof.* Each fibre of  $\pi$  outside infinity and the zero set of  $P_1(t) \dots P_r(t) = 0$  contains a geometrically integral irreducible component of multiplicity one. Since  $\pi$  has a section over the compositum  $K_1 \dots K_r$ , which is an abelian extension of  $\mathbb{Q}$ , assumptions (a) and (b) of Theorem 3.3 hold for  $\pi : X \rightarrow \mathbb{P}^1$ . By Hasse's norm theorem the varieties  $N_{K/\mathbb{Q}}(\mathbf{z}) = c$ , where  $K/\mathbb{Q}$  is cyclic and  $c \in \mathbb{Q}^*$ , satisfy the Hasse principle. Moreover, smooth and proper models of principal homogeneous spaces of cyclic norm tori satisfy the Hasse principle and weak approximation, by [San81, Ch. 8]. We conclude by Theorem 3.3.  $\square$

For  $r = 1$  the statement of Corollary 4.1 was previously known for any finite extension  $K/\mathbb{Q}$  in the case when  $P_1$  has at most two roots, see [HS02], [CHS03]; see also [SJ11], where  $\mathbb{Q}$  was replaced by an arbitrary number field.

For any cyclic extension of fields  $K/k$  the affine variety  $N_{K/k}(\mathbf{x}) = c$ , where  $c \in k^*$ , is well known to be birationally equivalent to the Severi–Brauer variety defined by the cyclic algebra  $(K/k, c)$ . Thus Corollary 4.1 can be seen as a particular case of Corollary 3.4.

When each polynomial  $P_i(t)$  is linear we have the following statement.

**Corollary 4.2.** *Let  $K_i$  be a cyclic extension of  $\mathbb{Q}$  of degree  $d_i$ ,  $i = 1, \dots, r$ . Let  $b_i \in \mathbb{Q}^*$  and  $e_i \in \mathbb{Q}$ ,  $i = 1, \dots, r$ . Then the variety over  $\mathbb{Q}$  defined by*

$$b_i(t - e_i) = N_{K_i/\mathbb{Q}}(\mathbf{x}_i) \neq 0, \quad i = 1, \dots, r,$$

*satisfies the Hasse principle and weak approximation.*

*Proof.* An easy calculation shows that this variety  $X$  is smooth. By Corollary 4.1 is enough to prove that  $\text{Br}_{\text{vert}}(X) = \text{Br}_0(X)$ . In Step 1 of the proof of Theorem 3.1 we saw that for any  $A \in \text{Br}(\mathbb{Q}(t))$  such that  $\pi^*A \in \text{Br}(X) \subset \text{Br}(\mathbb{Q}(X))$  there exists  $A_0 \in \text{Br}(\mathbb{Q})$  for which we can write

$$A = \sum_{i=1}^r n_i(K_i/\mathbb{Q}, t - e_i) + A_0.$$

Since  $(K_i/\mathbb{Q}, N_{K_i/\mathbb{Q}}(\mathbf{x}_i)) = 0$  in  $\text{Br}(\mathbb{Q}(X))$ , the element  $\pi^*A \in \text{Br}(X)$  can be written as

$$\pi^*A = - \sum_{i=1}^r n_i(K_i/\mathbb{Q}, b_i) + A_0 \in \text{Br}_0(X). \quad \square$$

The following statement is deduced from Corollary 4.1 by an easy application of the fibration method in the form of [H97, Thm. 3.2.1].

**Corollary 4.3.** *Let  $X$  be a smooth and proper model of the variety over  $\mathbb{Q}$  defined by the system of equations*

$$N_{K_i/\mathbb{Q}}(\mathbf{x}_i) = P_i(t_1, \dots, t_n), \quad i = 1, \dots, r, \quad (4.2)$$

*where each  $K_i$  is a cyclic extension of  $\mathbb{Q}$  and each  $P_i(t_1, \dots, t_n)$  is a product of polynomials of degree 1 over  $\mathbb{Q}$ . Then  $X(\mathbb{Q})$  is dense in  $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$ .*

In [ScSk12], under a mild general position condition, this was proved for  $r = 1$  and  $\deg P_1 \leq 2n$ , but with the cyclic extension  $K$  of  $\mathbb{Q}$  replaced by any finite extension of an arbitrary number field.

**4.2. Products of norms.** Instead of a norm form of a cyclic extension of  $\mathbb{Q}$  we can consider a product of norm forms associated to field extensions of  $\mathbb{Q}$  satisfying certain conditions. We start with one more application of Theorem 3.3.

**Corollary 4.4.** *Let  $P(t)$  be a product of (possibly repeated) linear factors over  $\mathbb{Q}$ . Let  $L_1, \dots, L_n$  be  $n \geq 2$  finite field extensions of  $\mathbb{Q}$  such that  $L_1/\mathbb{Q}$  is abelian and linearly disjoint from the compositum  $L_2 \dots L_n$ . Let  $X$  be a smooth, proper and geometrically integral variety over  $\mathbb{Q}$  with a morphism  $\pi : X \rightarrow \mathbb{P}^1$  such that the generic fibre of  $\pi$  is birationally equivalent to the affine variety*

$$N_{L_1/\mathbb{Q}}(\mathbf{x}_1) \dots N_{L_n/\mathbb{Q}}(\mathbf{x}_n) = P(t) \quad (4.3)$$

*over  $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$ . Then  $X$  satisfies the Hasse principle and weak approximation.*

*Proof.* This proof is similar to that of Corollary 4.1. Assumptions (a) and (b) of Theorem 3.3 are satisfied since  $L_1/\mathbb{Q}$  is abelian. To prove that almost all  $\mathbb{Q}$ -fibres satisfy the Hasse principle and weak approximation it is enough, by [San81, Ch. 8] (see also [Sk01, Thm. 6.3.1]), to verify that  $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}) = 0$ , where  $T$  is the multinorm torus over  $\mathbb{Q}$  attached to the fields  $L_1, \dots, L_n$ . This was proved by Demarche and Wei [DW12, Thm. 1]. To finish the proof we note that  $\text{Br}_{\text{vert}}(X) = \text{Br}_0(X)$ . Indeed, let  $A \in \text{Br}(\mathbb{Q}(t))$  be such that  $\pi^*A \in \text{Br}(X)$ . The morphism  $\pi$  has a section defined over  $L_i$ , for each  $i = 1, \dots, n$ . By restricting  $A$  to this section we see the restriction of  $A$  to  $\text{Br}(L_i(t))$  comes from  $\text{Br}(\mathbb{P}_{L_i}^1) = \text{Br}(L_i)$ . In particular, the residues of  $A$  at the roots of  $P(t)$  are in the kernel of the map  $H^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(L_i, \mathbb{Q}/\mathbb{Z})$ . Since  $L_1 \cap L_2 \dots L_n = \mathbb{Q}$  there is no non-trivial cyclic extension of  $\mathbb{Q}$  contained in all of the  $L_i$ . This implies that  $A$  is not ramified at the zero set of  $P(t)$ . A well known argument [CSw94, Prop. 1.1.1] shows that  $A$  is unramified away from the zero set of  $Q(t)$ . Hence  $A \in \text{Br}(\mathbb{A}^1) = \text{Br}(\mathbb{Q})$ .  $\square$

For more cases when  $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}) = 0$  for the multinorm torus  $T$  see [DW12], Thm. 1 and Cor. 8. One can extend Corollary 4.4 to systems of equations (4.3) and replace  $P(t)$  by a product of linear polynomials in several variables. We leave the details to the interested reader.

Following Wei [Wei12, Thm. 3.5] we now consider a case where the  $\mathbb{Q}$ -fibres do not satisfy the Hasse principle nor weak approximation.

**Proposition 4.5.** *Let  $P(t)$  be a product of (possibly repeated) linear factors over  $\mathbb{Q}$ , and let  $a, b \in \mathbb{Q}^*$ . Let  $X$  be a smooth, proper and geometrically integral variety over  $\mathbb{Q}$  with a morphism  $\pi : X \rightarrow \mathbb{P}^1$  such that the generic fibre of  $\pi$  is birationally equivalent to the affine variety*

$$N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x})N_{\mathbb{Q}(\sqrt{b})/\mathbb{Q}}(\mathbf{y})N_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}}(\mathbf{z}) = P(t) \quad (4.4)$$

over  $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(t)$ . Then  $X(\mathbb{Q})$  is dense in  $X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$ .

*Proof.* We can assume that  $\mathbb{Q}(\sqrt{a})$ ,  $\mathbb{Q}(\sqrt{b})$  and  $\mathbb{Q}(\sqrt{ab})$  are quadratic fields, otherwise the variety  $X$  is rational and the statement is clear. Let  $V$  be the smooth locus of the affine variety (4.4), and let  $U$  be the image of  $V$  by the projection to the coordinate  $t$ . It is clear that  $\mathbb{P}^1 \setminus U$  is a finite union of  $\mathbb{Q}$ -points. The fibres of  $V \rightarrow U$  are principal homogeneous spaces of the torus  $T$  that is given by

$$N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x})N_{\mathbb{Q}(\sqrt{b})/\mathbb{Q}}(\mathbf{y})N_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}}(\mathbf{z}) = 1.$$

Let  $E$  be a smooth equivariant compactification of  $T$  (which exists by [CHS05]), and let  $V^c = V \times^T E$  be the contracted product. Then  $V^c \rightarrow U$  is a fibre-wise smooth compactification of  $V \rightarrow U$ . We take  $\pi : X \rightarrow \mathbb{P}^1$  such that  $X \times_{\mathbb{P}^1} U = V^c$ . We compose  $\pi$  with an automorphism of  $\mathbb{P}^1$  to ensure that the fibre at infinity is smooth and is close to the real point that we need to approximate; in particular, the fibre at infinity contains a real point. An obvious change of variables shows that  $X$  contains an open set which is the smooth locus of the affine variety given by

$$N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x})N_{\mathbb{Q}(\sqrt{b})/\mathbb{Q}}(\mathbf{y})N_{\mathbb{Q}(\sqrt{ab})/\mathbb{Q}}(\mathbf{z}) = Q(t),$$

where  $Q(t)$  is a polynomial with rational roots  $e_1, \dots, e_r$  such that  $U$  is the complement to  $\{e_1, \dots, e_r\}$  in  $\mathbb{P}^1$ . Note that for any  $\tau \in U(\mathbb{Q})$  we have  $X_{\tau}(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$  by [C12, Prop. 5.1].

The quaternion algebra  $A = (N_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}}(\mathbf{x}), b)$  defines an element of  $\text{Br}(\pi^{-1}(U))$ .

We are given points  $M_p \in X(\mathbb{Q}_p)$  for all primes  $p$  and  $M_0 \in X(\mathbb{R})$  such that  $(M_p) \in X(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$ . Since  $\text{Br}(X)/\text{Br}_0(X)$  is finite, by a small deformation we can assume that  $\pi(M_p)$  is a point in  $U \cap \mathbb{A}^1$  where  $t$  equals  $\tau_p \in \mathbb{Q}_p$ .

Let  $S_0$  be the finite set of places of  $\mathbb{Q}$  where we need to approximate. We can find a finite set  $S$  of places containing  $S_0$  and the real place, such that  $\pi : X \rightarrow \mathbb{P}^1$  extends to a proper morphism  $\pi : \mathcal{X} \rightarrow \mathbb{P}_{\mathbb{Z}_S}^1$  with  $\mathcal{X}$  regular. By doing so we can ensure that  $S$  contains the primes where at least one of our quadratic fields is ramified, and that we have  $a, b \in \mathbb{Z}_S$ ,  $Q(t) \in \mathbb{Z}_S[t]$ , and  $e_i \in \mathbb{Z}_S$  for  $i = 1, \dots, r$ . By Harari's ‘formal lemma’ [H94, Cor. 2.6.1] we can further enlarge  $S$  so that

$$\sum_{p \in S} \text{inv}_p(A(M_p)) = 0, \quad \sum_{p \in S} \text{inv}_p(b, \tau_p - e_i) = 0, \quad i = 1, \dots, r.$$



(For this we may need to modify the points  $M_p$  for  $p \in S \setminus S_0$ .) Let  $\mathcal{U}$  be the complement to the Zariski closure of  $e_1 \cup \dots \cup e_r$  in  $\mathbb{P}_{\mathbb{Z}_S}^1$ . The same algebra  $A$  defines a class in  $\mathrm{Br}(\pi^{-1}(\mathcal{U}))$ . An application of Proposition 2.1 gives a  $\mathbb{Q}$ -point  $\tau$  in  $U \cap \mathbb{A}^1$  that is as large as we want in the real topology and is close to  $\tau_p$  in the  $p$ -adic topology for the primes  $p \in S$ . For  $p \notin S \cup \{p_1, \dots, p_r\}$  we see from property (3) of Proposition 2.1 that the Zariski closure of  $\tau$  in  $\mathbb{P}_{\mathbb{Z}_p}^1$  is contained in  $\mathcal{U} \times_{\mathbb{Z}_S} \mathbb{Z}_p$ . This implies that for any  $N_p \in X_\tau(\mathbb{Q}_p)$  the value  $A(N_p) \in \mathrm{Br}(\mathbb{Q}_p)$  comes from  $\mathrm{Br}(\mathbb{Z}_p) = 0$ . From property (5) we see that for each  $i = 1, \dots, r$  the prime  $p_i$  splits in  $\mathbb{Q}(\sqrt{b})$ , hence for any  $N_{p_i} \in X_\tau(\mathbb{Q}_{p_i})$  we have  $A(N_{p_i}) = 0$ . By continuity and the inverse function theorem we can find  $N_p \in X_\tau(\mathbb{Q}_p)$  close enough to  $M_p$ , for  $p \in S$  (including the real place), so that  $\sum_{p \in S} \mathrm{inv}_p(A(N_p)) = 0$ . Summing over all places of  $\mathbb{Q}$  we now have  $\sum_p \mathrm{inv}_p(A(N_p)) = 0$ , for any choice of  $N_p$ ,  $p \notin S$ . By [C12, Thm. 4.1] the algebra  $A$  generates  $\mathrm{Br}(X_\tau)$  modulo the image of  $\mathrm{Br}(\mathbb{Q})$ . By [San81, Ch. 8] or [Sk01, Thm. 6.3.1] the set  $X_\tau(\mathbb{Q})$  is dense in  $X_\tau(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$ , so we can find a  $\mathbb{Q}$ -point in  $X_\tau$  close to  $M_p$ , for  $p \in S$ .  $\square$

**4.3. Non-cyclic extensions of prime degree.** The method of Colliot-Thélène and Wei [Wei12, Thm. 3.6] can be used to prove the following result.

**Theorem 4.6.** *Let  $P(t)$  be a product of (possibly repeated) linear factors over  $\mathbb{Q}$ . Let  $K$  be a non-cyclic extension of  $\mathbb{Q}$  of prime degree such that the Galois group of the normal closure of  $K$  over  $\mathbb{Q}$  has a non-trivial abelian quotient. Let  $X$  be a smooth, proper and geometrically integral variety over  $\mathbb{Q}$  with a morphism  $\pi : X \rightarrow \mathbb{P}^1$  such that the generic fibre of  $\pi$  is birationally equivalent to the affine variety  $N_{K/\mathbb{Q}}(\mathbf{x}) = P(t)$  over  $\mathbb{Q}(t)$ . Then  $X$  satisfies the Hasse principle and weak approximation.*

This result covers the ‘generic’ case of the field  $K = \mathbb{Q}[t]/(f(t))$ , for which  $f(t)$  is a polynomial of prime degree  $\ell > 2$  such that the Galois group of  $f(t)$  is the symmetric group  $S_\ell$  (or the dihedral group  $D_{2\ell}$ .) In particular, the assumption on  $K$  in Theorem 4.6 is automatically satisfied when  $[K : \mathbb{Q}] = 3$ .

*Proof of Theorem 4.6.* We start in the same way as in the proof of Proposition 4.5. We can assume that  $X$  contains an open set which is the smooth locus of the affine variety  $N_{K/\mathbb{Q}}(\mathbf{x}) = Q(t)$ , where  $Q(t)$  is a product of powers of  $t - e_i$ ,  $i = 1, \dots, e_r$ , with the additional assumption that the fibre at infinity is smooth and contains a real point close to the real point that we want to approximate.

**Lemma 4.7.** *We have  $\mathrm{Br}(X) = \mathrm{Br}_0(X)$ .*

*Proof.* Let  $T$  be the norm torus  $N_{K/\mathbb{Q}}(\mathbf{x}) = 1$ . Since  $\ell = [K : \mathbb{Q}]$  is prime, [CSan87a, Prop. 9.1, Prop. 9.5] gives  $H^1(F, \mathrm{Pic}(Z \times_F \overline{F})) = \mathrm{III}_\omega^2(F, \hat{T}) = 0$  for any smooth and proper variety  $Z$  over a field  $F$  such that a dense open subset of  $Z$  is a principal homogeneous space of  $T$ . Applying this to the generic fibre of  $\pi : X \rightarrow \mathbb{P}^1$  we see that  $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X)$ .

Now let  $A \in \text{Br}(\mathbb{Q}(t))$  be such that  $\pi^*A \in \text{Br}(X)$ . The morphism  $\pi$  has a section defined over  $K$ . By restricting to it we see that the image of  $A$  in  $\text{Br}(K(t))$  belongs to the injective image of  $\text{Br}(\mathbb{P}_K^1) = \text{Br}(K)$ . In particular, the residue of  $A$  at  $e_i$  lies in the kernel of the map

$$H^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(K, \mathbb{Q}/\mathbb{Z}).$$

Since  $K$  contains no cyclic extension of  $\mathbb{Q}$ , this kernel is zero. Thus  $A$  is not ramified at the zero set of  $Q(t)$ . Since  $A$  is also unramified outside of the zero set of  $Q(t)$ , we see that  $A \in \text{Br}(\mathbb{Q})$ .  $\square$

Let  $L$  be the normal closure of  $K/\mathbb{Q}$ . By assumption there exists a cyclic extension  $k/\mathbb{Q}$  of prime degree such that  $k \subset L$ . Let  $\ell = [K : \mathbb{Q}]$ ,  $q = [k : \mathbb{Q}]$ . Since  $\text{Gal}(L/\mathbb{Q}) \subset S_\ell$  and  $k \neq K$  we see that  $q < \ell$ .

**Lemma 4.8.** *Let  $a \in \mathbb{Q}^*$ . If  $p$  is a prime unramified in  $L$  and inert in  $k$ , then the equation  $N_{K/\mathbb{Q}}(\mathbf{x}) = a$  is solvable in  $\mathbb{Q}_p$ .*

*Proof.* Write  $K \otimes_{\mathbb{Q}} \mathbb{Q}_p = K_{v_1} \oplus \dots \oplus K_{v_s}$ , and let  $d_i = [K_{v_i} : \mathbb{Q}_p]$ .

If  $s > 1$ , then since  $\ell = d_1 + \dots + d_s$  is a prime number, there exist integers  $n_1, \dots, n_s$  such that  $1 = n_1 d_1 + \dots + n_s d_s$ . It follows that

$$a = \prod_{i=1}^s N_{K_{v_i}/\mathbb{Q}_p}(a^{n_i}) \in N_{K/\mathbb{Q}}(K \otimes_{\mathbb{Q}} \mathbb{Q}_p),$$

so we are done.

If  $s = 1$ , then  $K \otimes_{\mathbb{Q}} \mathbb{Q}_p = K_v$  is a field extension of  $\mathbb{Q}_p$  of degree  $\ell$ . By assumption  $p$  is inert in  $k$ , so that  $k \otimes_{\mathbb{Q}} \mathbb{Q}_p = k_w$  is a field. Since  $[k_w : \mathbb{Q}_p] = q$  is a prime less than  $\ell$ , the fields  $k_w$  and  $K_w$  are linearly disjoint over  $\mathbb{Q}_p$ , so that  $k_w K_v$  is a field. Thus  $p$  is inert in  $kK \subset L$ , which implies that the Frobenius at  $p$  in  $\text{Gal}(L/\mathbb{Q})$  is an element of order divisible by  $\ell q$ . However,  $S_\ell$  contains no such elements, so the case  $s = 1$  is impossible.  $\square$

*End of proof of Theorem 4.6.* We are given points  $M_p \in X(\mathbb{Q}_p)$  for all primes  $p$  and  $M_0 \in X(\mathbb{R})$ . By a small deformation we can assume that  $\pi(M_p)$  is a point in  $U \cap \mathbb{A}^1$  where  $t$  equals  $\tau_p \in \mathbb{Q}_p$ . Let  $S$  be the finite set of places of  $\mathbb{Q}$  where we need to approximate, containing the real place and the primes of bad reduction for  $X$ . We also assume that  $L$  is unramified away from  $S$ . Consider the cyclic algebras

$$A_i = (k/\mathbb{Q}, t - e_i) \in \text{Br}(\mathbb{Q}(X)), \quad i = 1, \dots, r.$$

Harari's ‘formal lemma’ [H94, Cor. 2.6.1] and Lemma 4.7 imply that we can introduce new primes into  $S$  and choose the corresponding points  $M_p$  so that

$$\sum_{p \in S} \text{inv}_p(A_i(\tau_p)) \neq 0, \quad i = 1, \dots, r.$$

An application of Proposition 2.1 gives a  $\mathbb{Q}$ -point  $\tau$  in  $U \cap \mathbb{A}^1$  that is as large as we want in the real topology and is close to  $\tau_p$  in the  $p$ -adic topology for the primes  $p \in S$ . This ensures that  $X_\tau(\mathbb{R}) \neq \emptyset$  and  $X_\tau(\mathbb{Q}_p) \neq \emptyset$  for all  $p \in S$ . For  $p \notin S \cup \{p_1, \dots, p_r\}$  we see from property (3) of Proposition 2.1 that  $\tau$  reduces

modulo  $p$  to a point of  $\mathbb{P}^1(\mathbb{F}_p)$  other than the reduction of any of  $e_1, \dots, e_r$ . The corresponding fibre over  $\mathbb{F}_p$  contains a principal homogeneous space of a torus over a finite field, and hence an  $\mathbb{F}_p$ -point, by Lang’s theorem. By Hensel’s lemma it gives rise to a  $\mathbb{Q}_p$ -point in  $X_\tau$ . Finally, property (5) of Proposition 2.1 gives that  $\text{inv}_{p_i}(A_i(\tau)) \neq 0$ . By property (4) this implies that  $p_i$  is inert in  $k$ . Now an application of Lemma 4.8 shows that  $X_\tau(\mathbb{Q}_{p_i}) \neq \emptyset$ . This holds for all  $i = 1, \dots, r$  so we conclude that  $X_\tau(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ . To finish the proof we note that  $\text{III}_{\omega}^2(\mathbb{Q}, \hat{T}) = 0$  implies that the principal homogeneous spaces of  $T$  over  $\mathbb{Q}$  satisfy the Hasse principle and weak approximation [San81, Ch. 8].  $\square$

## 5. PENCILS OF CURVES OF GENUS 1

Swinnerton-Dyer [Sw95], and later Swinnerton-Dyer, Colliot-Thélène and one of the authors [CSkSw98b] applied Hypothesis  $(H_1)$  to pencils of curves of genus 1 (see [W07] for a general exposition of this method and other applications). As we explained in Section 2, Hypothesis  $(H_1)$  can be replaced by Proposition 2.1 as long as all the singular fibres of the pencil are defined over  $\mathbb{Q}$ . We now revisit some of the surfaces to which the main theorem of [CSkSw98b] applies and give examples satisfying this condition. Since the theorem in [CSkSw98b, p. 583] assumes the finiteness of the Tate–Shafarevich group of elliptic curves, the main results in this section are conditional on a certain finiteness property of the Tate–Shafarevich groups of elliptic curves over  $\mathbb{Q}$ .

Let us consider the ‘Egloffstein case’ [CSkSw98b, Section 3.2] which goes back to [Sw95], with additional constraints ensuring that all singular fibres are  $\mathbb{Q}$ -fibres. Let  $\alpha_i, \gamma_i \in \mathbb{Q}^*$  for  $i = 1, 2, 3$  be such that  $\gamma_i \neq \pm\gamma_j$  when  $i \neq j$ . Then the intersection of two quadrics  $W \subset \mathbb{P}^4$

$$\alpha_1\gamma_1^2x_1^2 + \alpha_2\gamma_2^2x_2^2 + \alpha_3\gamma_3^2x_3^2 + x_4^2 = x_0^2 + \alpha_1x_1^2 + \alpha_2x_2^2 + \alpha_3x_3^2 = 0.$$

is smooth, and so is a del Pezzo surface of degree 4. Exactly five quadrics in the pencil of quadrics containing  $W$  have rank 4. Their discriminants, defined up to a square, are

$$\varepsilon_0 = \varepsilon_4 = \alpha_1\alpha_2\alpha_3, \quad \varepsilon_i = -\alpha_j\alpha_k(\gamma_i^2 - \gamma_j^2)(\gamma_i^2 - \gamma_k^2),$$

where  $\{i, j, k\} = \{1, 2, 3\}$ . Note that  $\varepsilon_1\varepsilon_2\varepsilon_3 = 1 \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ .

**Theorem 5.1.** *If  $\varepsilon_0, \varepsilon_1, \varepsilon_2$  are linearly independent in the  $\mathbb{F}_2$ -vector space  $\mathbb{Q}^*/\mathbb{Q}^{*2}$ , then  $\text{Pic}(W) \simeq \mathbb{Z}$  and  $\text{Br}(W)/\text{Br}_0(W) \simeq \mathbb{Z}/2$ , so that  $W$  is a minimal del Pezzo surface of degree 4 without a conic bundle structure. Assume the finiteness of the 2-primary torsion subgroup of the Tate–Shafarevich group of any elliptic curve of Mordell–Weil rank 0 over  $\mathbb{Q}$ . Then  $W(\mathbb{Q})$  is dense in  $W(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$ .*

*Proof.* The computation of  $\text{Pic}(W)$  and  $\text{Br}(W)/\text{Br}_0(W)$  is well known, for the latter see, e.g., [W07, Thm. 3.37]. Let  $X$  be the blowing-up of  $W$  in the four points  $x_0 = x_4 = 0$ . Let  $\pi : X \rightarrow \mathbb{P}^1$  be the morphism such that the fibre

at  $(\mu : \nu)$  is the intersection of  $W$  with the hyperplane  $\mu x_0 = \nu x_4$ :

$$t^2 x_0^2 + \alpha_1 \gamma_1^2 x_1^2 + \alpha_2 \gamma_2^2 x_2^2 + \alpha_3 \gamma_3^2 x_3^2 = x_0^2 + \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 0,$$

where  $t = \mu/\nu$ . The fibre at  $(1 : 0)$  is smooth, and the singular geometric fibres of  $\pi$  are given by  $t = \pm \gamma_i$ , thus all singular fibres of  $\pi$  are defined over  $\mathbb{Q}$  and there are six of them. Every smooth fibre of  $\pi$  is a curve of genus 1 in  $\mathbb{P}^3$  given by two simultaneously diagonal quadrics, so it is a 2-covering of its Jacobian, which has rational 2-division points. The surface  $W$  is a particular case of the situation considered in Section 3.1 of [CSkSw98b]. Thus in order to apply the main theorem of [CSkSw98b], with Proposition 2.1 replacing Hypothesis (H<sub>1</sub>), it is enough to check that condition (D) holds. The statement (a) of the theorem in [CSkSw98b, p. 583] then gives  $W(\mathbb{Q}) \neq \emptyset$ . For a del Pezzo surface of degree 4 this is known to imply the density of  $W(\mathbb{Q})$  in  $W(\mathbf{A}_{\mathbb{Q}})^{\text{Br}}$ , by the main result of [SaSk91]. Finally, the finiteness of the Tate–Shafarevich group is used in the proof of the main theorem of [CSkSw98b] only for the 2-primary torsion subgroup and only for elliptic curves of Mordell–Weil rank 0 (“The final step”, p. 617).

We now check that condition (D) holds following the recipe given in [CSkSw98b]. Let  $U = \mathbb{P}^1 \setminus \{\pm \gamma_1, \pm \gamma_2, \pm \gamma_3\}$ . In the notation of [CSkSw98b] the group  $\mathfrak{S}$  is the subgroup of  $(\mathbb{Q}[U]^*/\mathbb{Q}[U]^{*2})^2$  generated by  $(\mathbb{Q}^*/\mathbb{Q}^{*2})^2$  and the six elements

$$\sigma_1^{\pm} = (1, t \pm \gamma_1), \quad \sigma_2^{\pm} = (t \pm \gamma_2, 1), \quad \sigma_3^{\pm} = (t \pm \gamma_3, t \pm \gamma_3).$$

The calculation of the homomorphism

$$\delta = (\delta_{\gamma_1}, \delta_{-\gamma_1}, \delta_{\gamma_2}, \delta_{-\gamma_2}, \delta_{\gamma_3}, \delta_{-\gamma_3}) : \mathfrak{S} \longrightarrow (\mathbb{Q}^*/\mathbb{Q}^{*2})^6$$

in [CSkSw98b, Lemma 2.2.1] gives the following explicit description. The element  $(b_1, b_2) \in (\mathbb{Q}^*/\mathbb{Q}^{*2})^2$  is sent by  $\delta$  to  $(b_1, b_1, b_2, b_2, b_1 b_2, b_1 b_2)$ . Next,  $\delta$  sends  $\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-, \sigma_3^+, \sigma_3^-$  (in this order) to the rows of the following matrix:

$$\begin{pmatrix} 1 & -\alpha_1 \varepsilon_1 & \gamma_1 + \gamma_2 & \gamma_1 - \gamma_2 & \gamma_1 + \gamma_3 & \gamma_1 - \gamma_3 \\ -\alpha_1 \varepsilon_1 & 1 & -\gamma_1 + \gamma_2 & -\gamma_1 - \gamma_2 & -\gamma_1 + \gamma_3 & -\gamma_1 - \gamma_3 \\ \gamma_1 + \gamma_2 & -\gamma_1 + \gamma_2 & 1 & -\alpha_2 \varepsilon_2 & \gamma_2 + \gamma_3 & \gamma_2 - \gamma_3 \\ \gamma_1 - \gamma_2 & -\gamma_1 - \gamma_2 & -\alpha_2 \varepsilon_2 & 1 & -\gamma_2 + \gamma_3 & -\gamma_2 - \gamma_3 \\ \gamma_1 + \gamma_3 & -\gamma_1 + \gamma_3 & \gamma_2 + \gamma_3 & -\gamma_2 + \gamma_3 & 1 & -\alpha_3 \varepsilon_3 \\ \gamma_1 - \gamma_3 & -\gamma_1 - \gamma_3 & \gamma_2 - \gamma_3 & -\gamma_2 - \gamma_3 & -\alpha_3 \varepsilon_3 & 1 \end{pmatrix}$$

The subgroup  $\mathfrak{S}_0 \subset \mathfrak{S}$  consists of the elements both coordinates of which have even degree. The class  $\mathbf{m} = (m_1, m_2)$  is  $(\varepsilon_1, \varepsilon_2)$ . Condition D says that the kernel of the composed map

$$\mathfrak{S}_0 \xrightarrow{\delta} (\mathbb{Q}^*/\mathbb{Q}^{*2})^6 \longrightarrow \mathbb{Q}^*/\langle \mathbb{Q}^{*2}, \delta_{\gamma_1}(\mathbf{m}) \rangle \oplus \cdots \oplus \mathbb{Q}^*/\langle \mathbb{Q}^{*2}, \delta_{-\gamma_3}(\mathbf{m}) \rangle$$

has dimension 3, and is generated by  $\mathbf{m}$  and by the classes of the points of order 2 of the Jacobian of the generic fibre of  $\pi$ . The latter classes may be written as  $\tau_1 \sigma_2^+ \sigma_2^- \sigma_3^+ \sigma_3^-$ ,  $\tau_2 \sigma_1^+ \sigma_1^- \sigma_3^+ \sigma_3^-$  and  $\tau_3 \sigma_1^+ \sigma_1^- \sigma_2^+ \sigma_2^-$  for some  $\tau_1, \tau_2, \tau_3 \in (\mathbb{Q}^*/\mathbb{Q}^{*2})^2$ . Thus, in order to check Condition D, it is enough to consider  $\xi = (\sigma_1^+)^m (\sigma_2^+)^{m'} (\sigma_3^+)^{\ell} (\sigma_3^-)^{\ell'} (c, d)$ , where  $m, m', \ell, \ell' \in \mathbb{F}_2$  and  $c, d \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ . We have  $\xi \in \mathfrak{S}_0$  if and only if  $m' = m = \ell' + \ell$ . The conditions  $\delta_{\gamma_1}(\xi) = \varepsilon_1^{\alpha_1}$

and  $\delta_{-\gamma_1}(\xi) = \varepsilon_1^{a'_1}$  imply  $\varepsilon_0^m = \varepsilon_1^{a_1+a'_1}$ . Thanks to our assumption we deduce  $m = 0$ , hence  $\xi = (\sigma_3^+ \sigma_3^-)^\ell(c, d)$ . By calculating  $\delta_{\gamma_1}(\xi)$ ,  $\delta_{\gamma_2}(\xi)$  and  $\delta_{\gamma_3}(\xi)$  we obtain

$$(\gamma_1^2 - \gamma_3^2)^\ell c = \varepsilon_1^{a_1}, (\gamma_2^2 - \gamma_3^2)^\ell d = \varepsilon_2^{a_2}, (-\alpha_3 \varepsilon_1 \varepsilon_2)^\ell cd = (\varepsilon_1 \varepsilon_2)^{a_3}.$$

Multiplying these identities we find  $\varepsilon_0^\ell = \varepsilon_1^{a_1+a_3} \varepsilon_2^{a_2+a_3}$ , which by our assumption implies  $\ell = 0$  and  $a_1 = a_2 = a_3$ , so that  $\xi = 1$  or  $\xi = (\varepsilon_1, \varepsilon_2) = \mathbf{m}$ . Thus condition (D) holds.  $\square$

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